# Bandlets and Model Selection for Image Estimation

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F. Autin, J.-M. Loubes et V. Rivoirard

5 September 2007

# Outline

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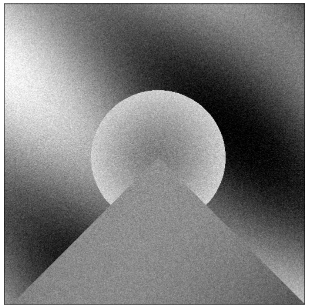
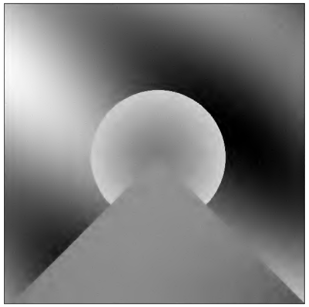
Noisy

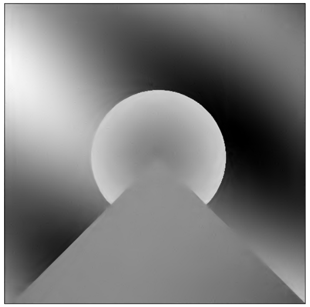


Estimation

Wavelets



Bandlets



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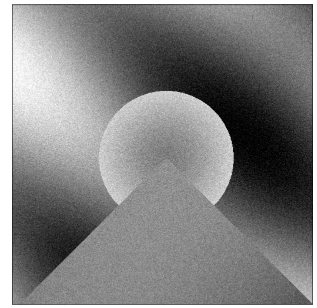
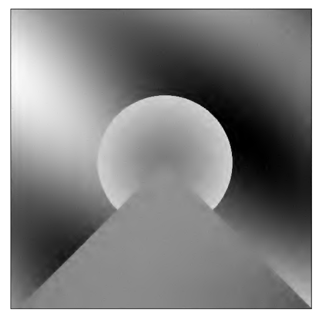
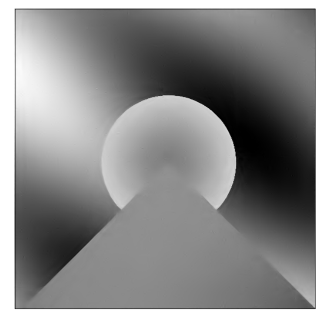


Estimation Wavelets Bandlets





- Geometric image approximation.

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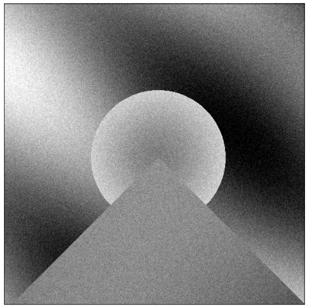
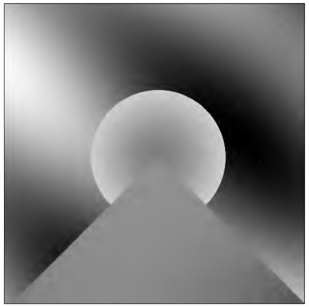
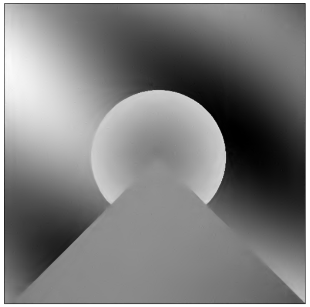


Estimation  Wavelets  Bandlets

- Geometric image approximation.
- Bandlet basis and bandlet estimation.

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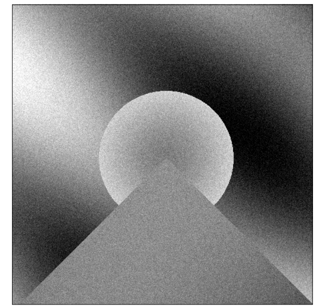

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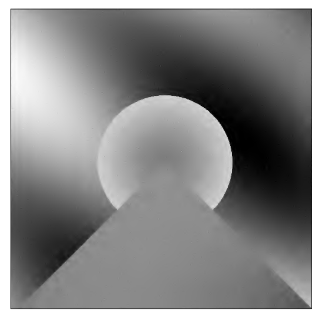

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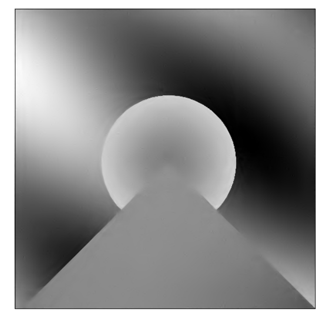

- Geometric image approximation.
- Bandlet basis and bandlet estimation.
- Thresholding, model selection, minimax and maxiset.

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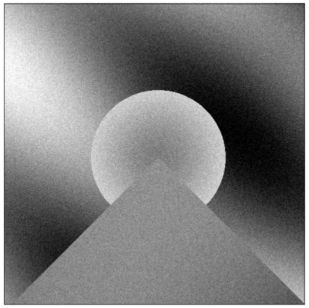
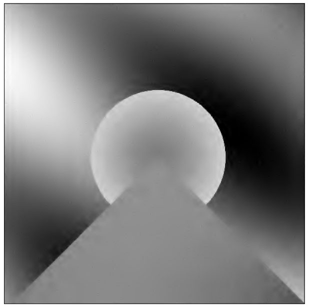
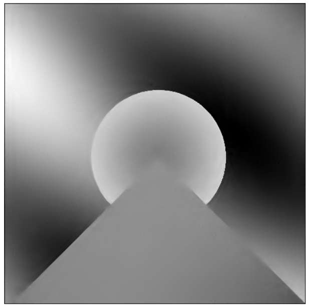


Estimation Wavelets Bandlets





- Geometric image approximation.
- Bandlet basis and bandlet estimation.
- Thresholding, model selection, minimax and maxiset.
- Sketch of proof.

# Geometric Image and Wavelet Approximation

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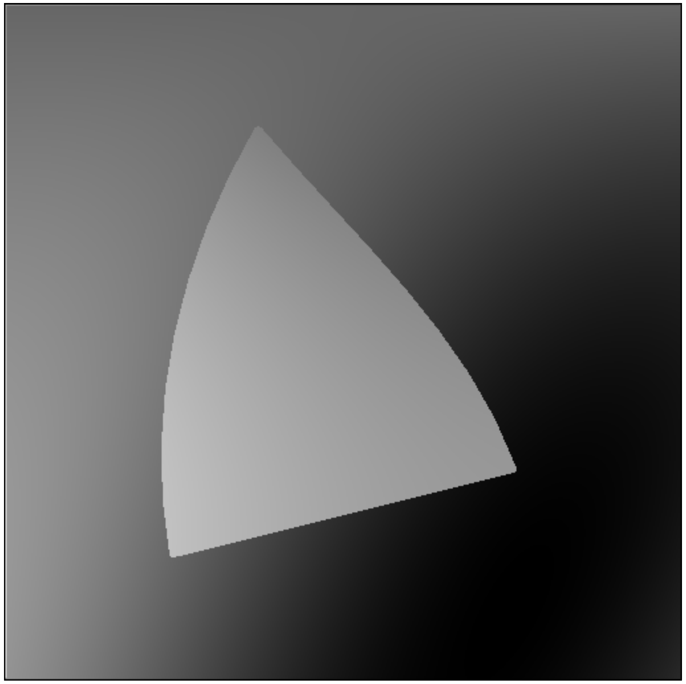
- Approximations of $f$ which is $\mathbf{C}^\alpha$ away from $\mathbf{C}^\alpha$ "edge" curves:



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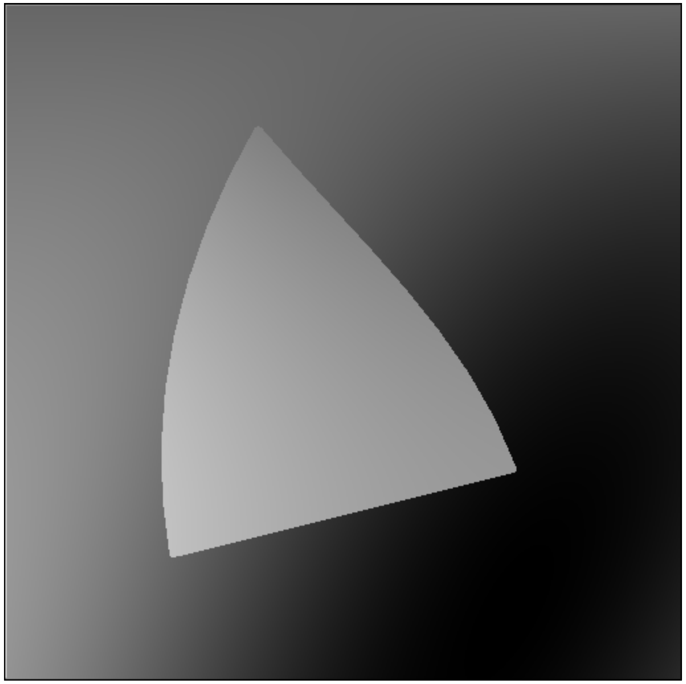

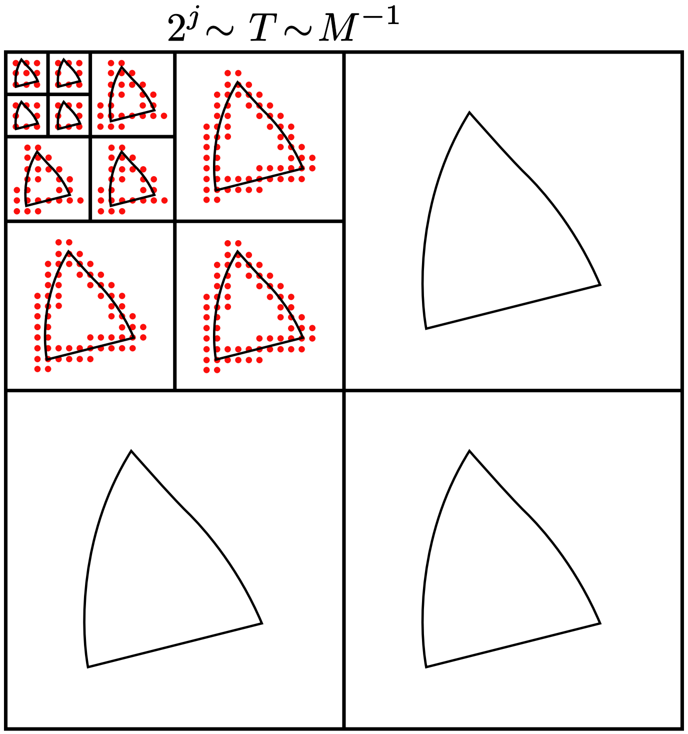


- With $M$ wavelets: $\|f - f_M\|^2 \leq C\, M^{-1}$.

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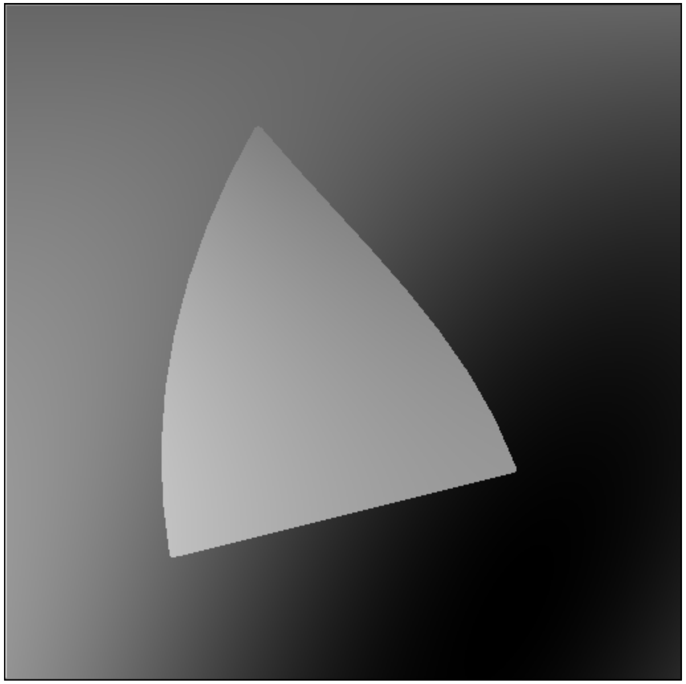

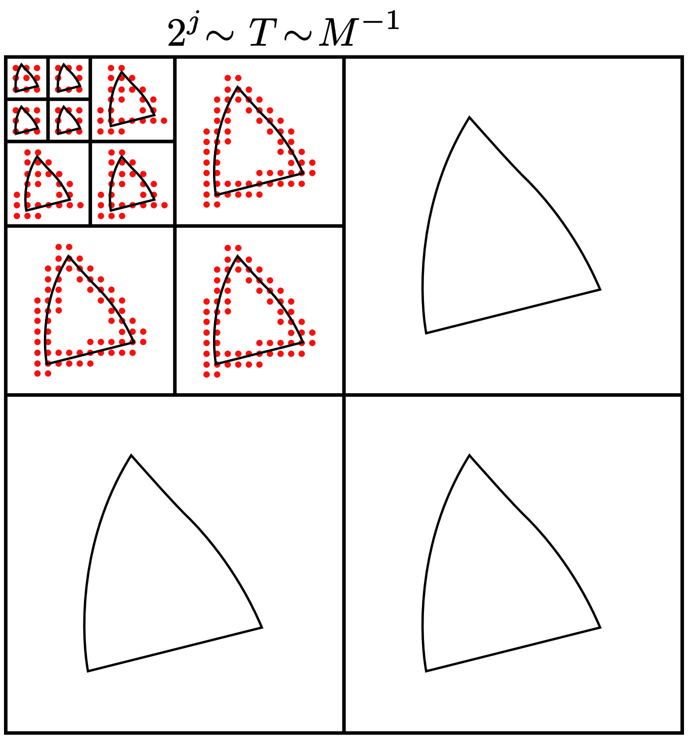


- With $M$ wavelets: $\|f - f_M\|^2 \leq C\, M^{-1}$.
- (*Cohen, DeVore, Petrushev, Xue*): Optimal for bounded variation functions: $\|f - f_M\|^2 \leq C\, M^{-1}$.

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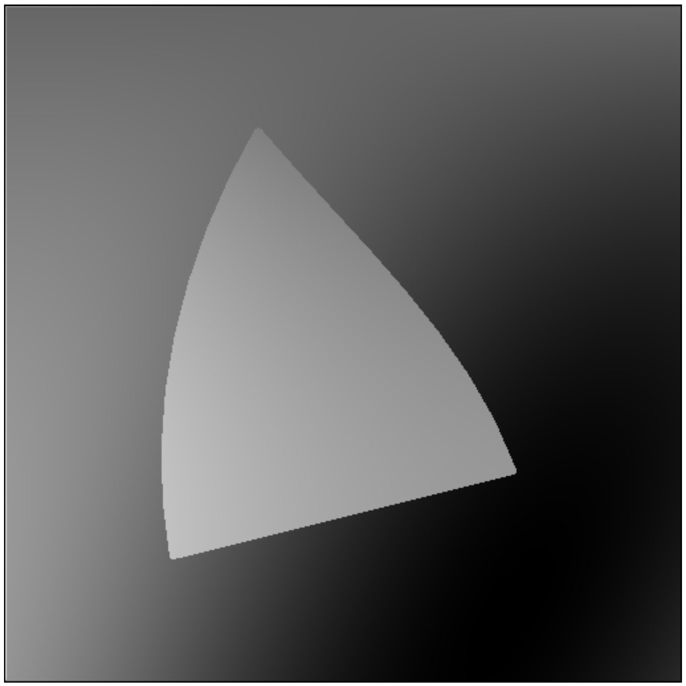

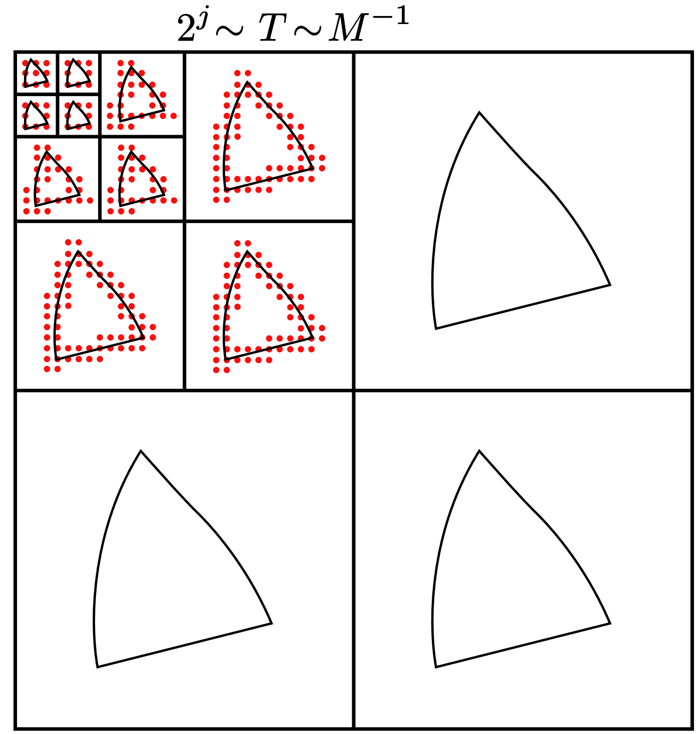


- With $M$ wavelets: $\|f - f_M\|^2 \leq C\, M^{-1}$.
- (*Cohen, DeVore, Petrushev, Xue*): Optimal for bounded variation functions: $\|f - f_M\|^2 \leq C\, M^{-1}$.
- But: does not take advantage of any geometric regularity.

# Geometric Elements for Contours

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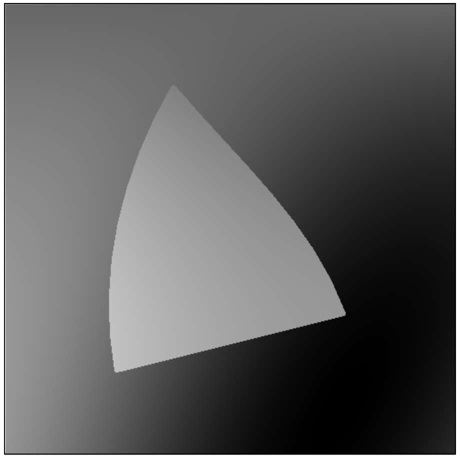
- Approximation of $f$ that is $\mathbf{C}^\alpha$ outside $\mathbf{C}^\alpha$ contours:



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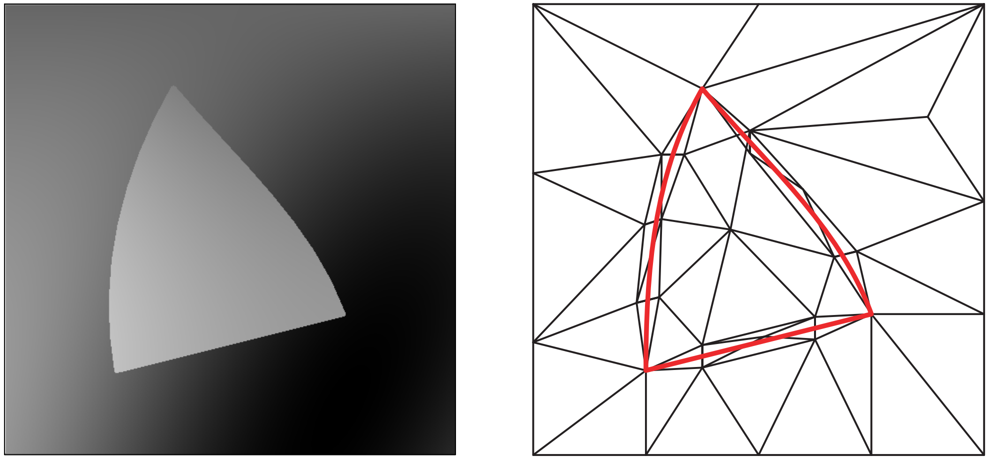
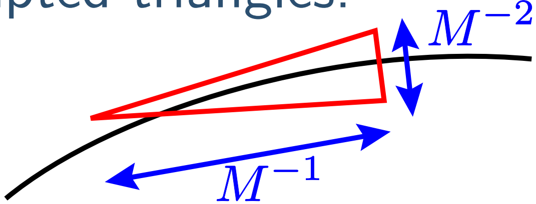


- Piecewise linear approximation with $M$ adapted triangles:
  if $\alpha \geq 2$ then $\|f - f_M\|^2 \leq C\, M^{-2}$.



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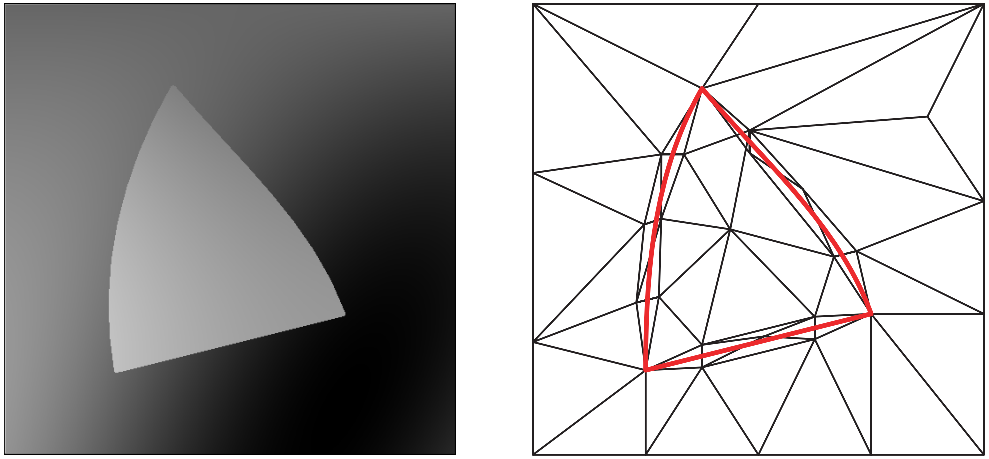

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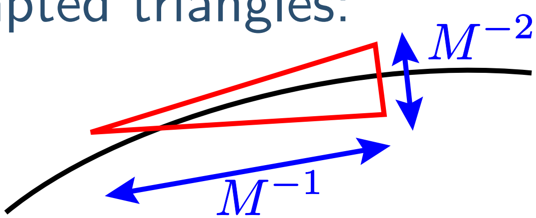
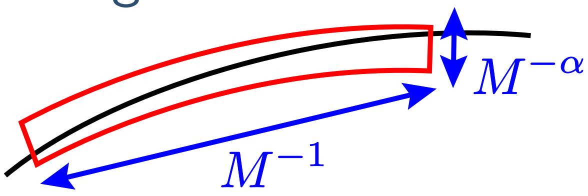



- Higher order approximation with $M$ adapted "higher order elements":
  $\|f - f_M\|^2 \leq C\, M^{-\alpha}$.



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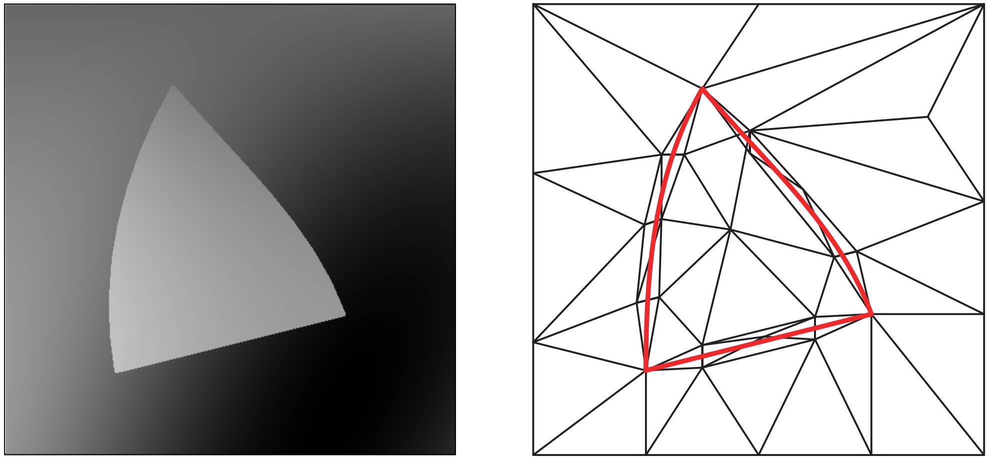

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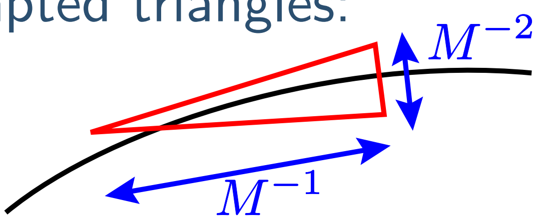


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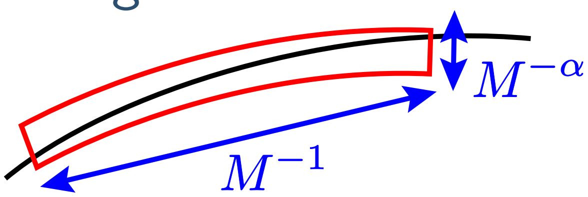


- Not a basis and difficult optimization.

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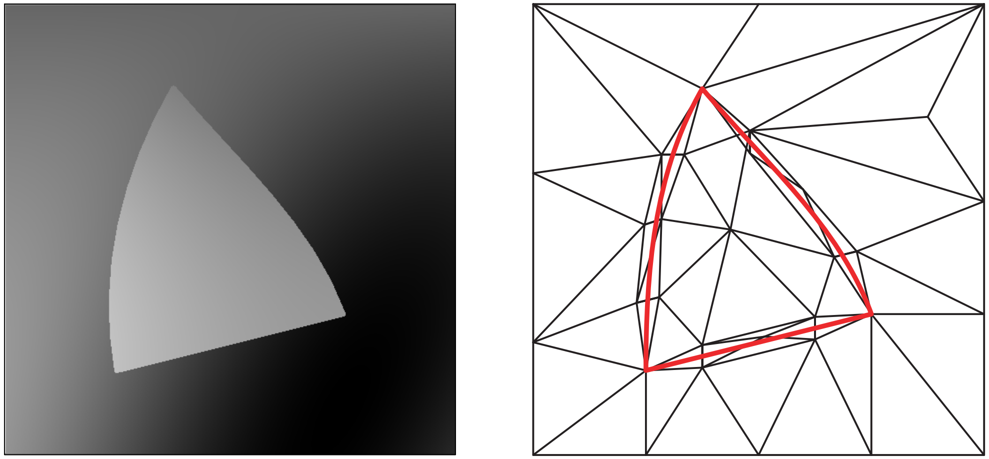

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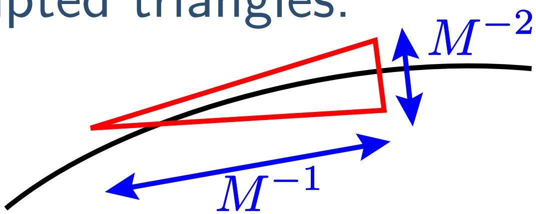


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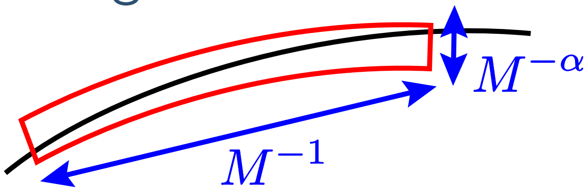


- Not a basis and difficult optimization.
- If $\alpha = 2$ : curvelet tight frame is almost optimal.

# Bandlets

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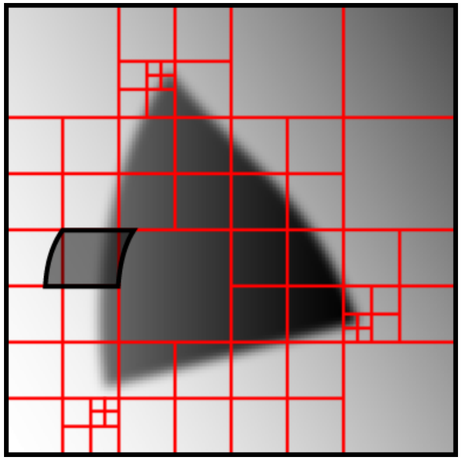

- Image $\mathbf{C}^{\alpha} - \mathbf{C}^{\alpha}$ simple

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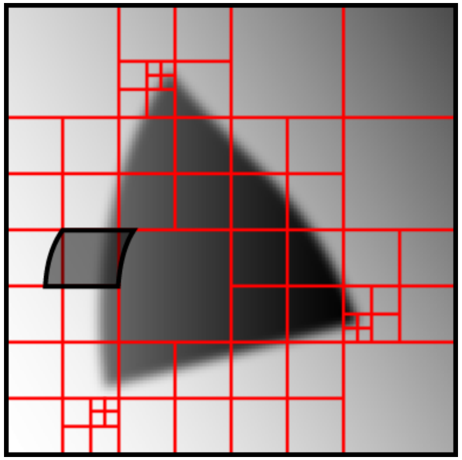

- Image $\mathbf{C}^{\alpha} - \mathbf{C}^{\alpha}$ simple piecewise.

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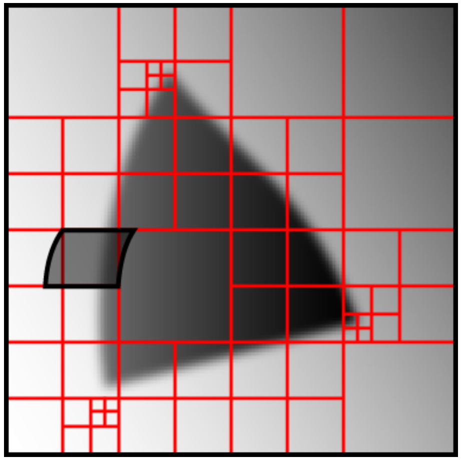

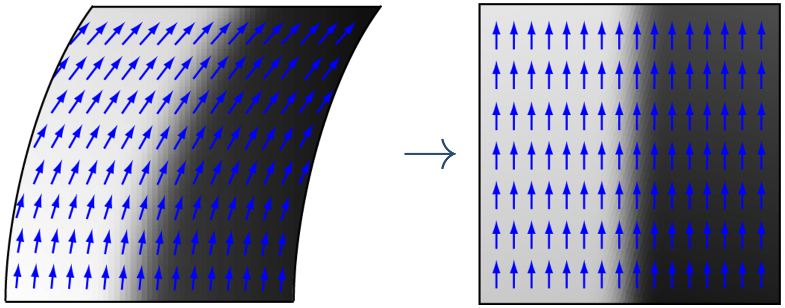

- Image $\mathbf{C}^\alpha - \mathbf{C}^\alpha$ simple piecewise.
- Locale deformation $\implies$ vertical/horizontal singularities.

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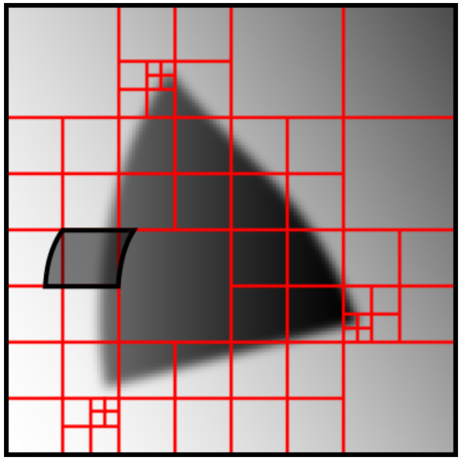

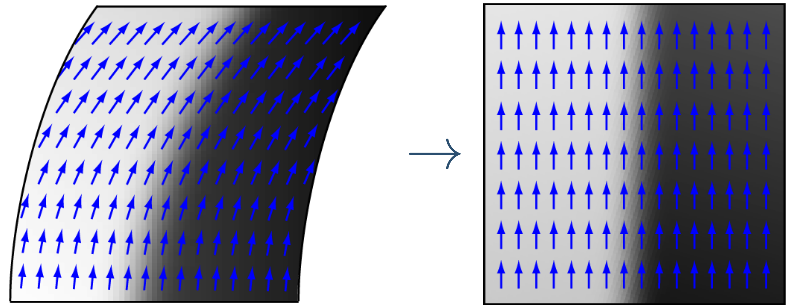

- Image $\mathbf{C}^\alpha - \mathbf{C}^\alpha$ simple piecewise.
- Locale deformation $\Longrightarrow$ vertical/horizontal singularities.
- Local bandlets: preimage of an adapted basis.

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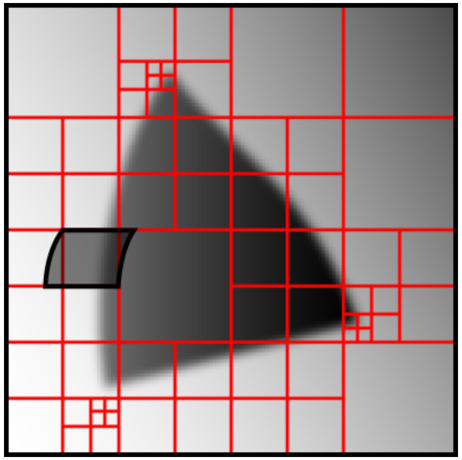

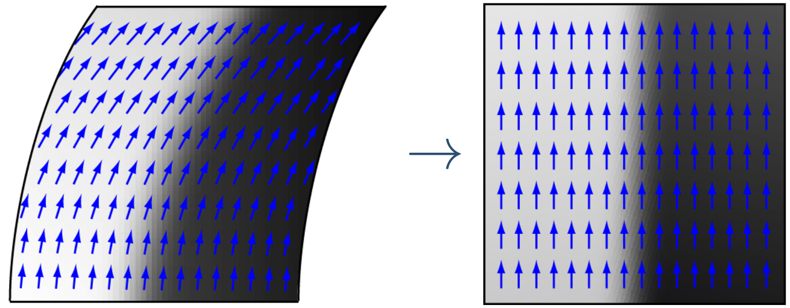

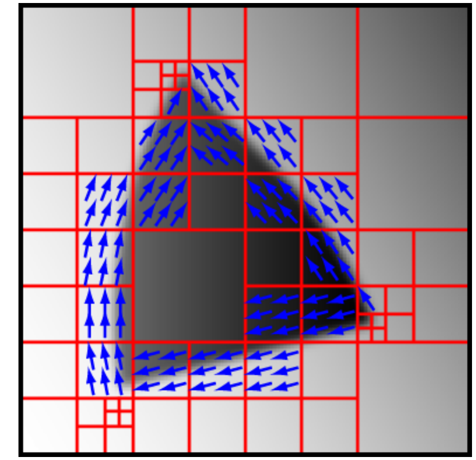

- Image $\mathbf{C}^\alpha - \mathbf{C}^\alpha$ simple piecewise.
- Locale deformation $\Longrightarrow$ vertical/horizontal singularities.
- Local bandlets: preimage of an adapted basis.
- Bandlet basis: dyadic segmentation + a geometry per square.

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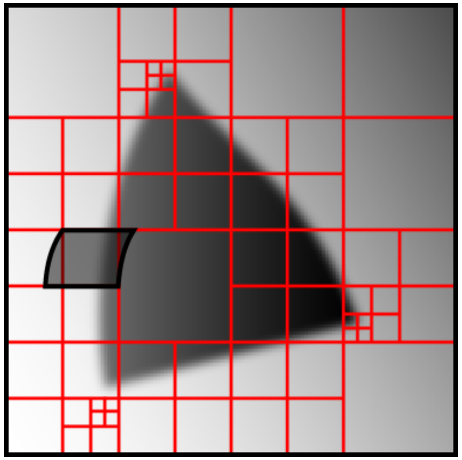

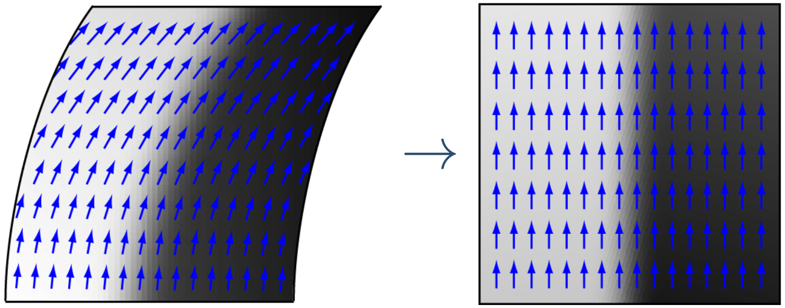

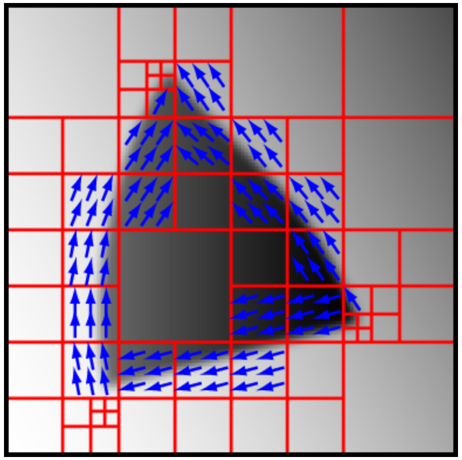

- Image $\mathbf{C}^{\alpha} - \mathbf{C}^{\alpha}$ simple piecewise.
- Locale deformation $\Longrightarrow$ vertical/horizontal singularities.
- Local bandlets: preimage of an adapted basis.
- Bandlet basis: dyadic segmentation + a geometry per square.
- **Theorem:** If $f$ is $\mathbf{C}^{\alpha} - \mathbf{C}^{\alpha}$, then, in a best basis,

$$\|f - f_M\|^2 \leq C(\log M) M^{-\alpha} \quad .$$

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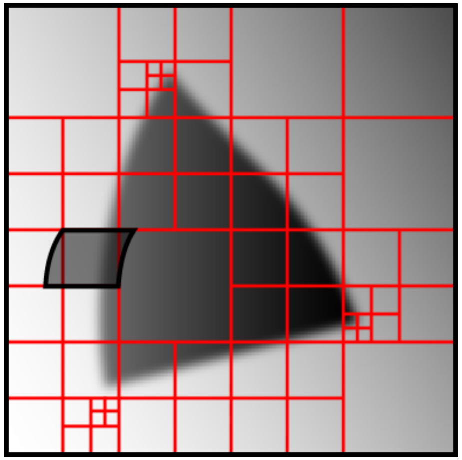

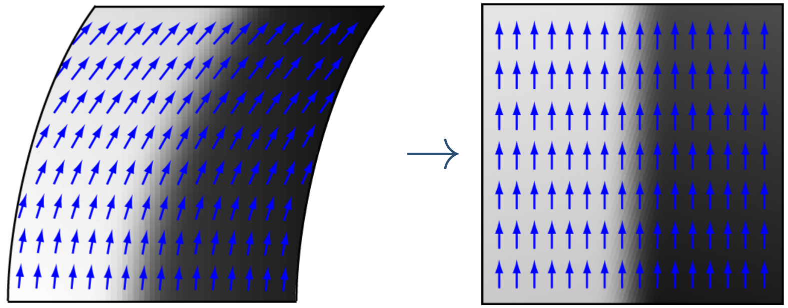

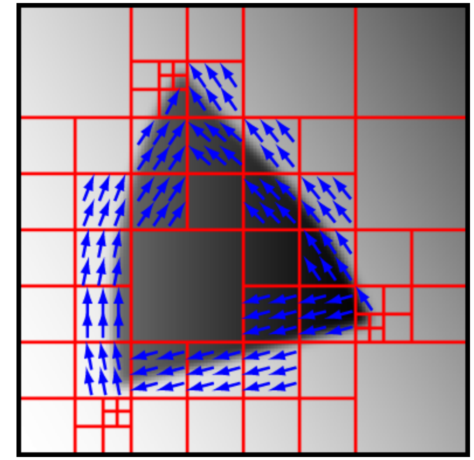

- Image $\mathbf{C}^\alpha - \mathbf{C}^\alpha$ simple piecewise.
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- **Theorem:** If $f$ is $\mathbf{C}^\alpha - \mathbf{C}^\alpha$, then, in a best basis,

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- Lagrangian approach: minimization of $\|f - f_M\|^2 + T^2 M \Longrightarrow$ thresholding in a fixed basis (easy) and best basis search (difficult).

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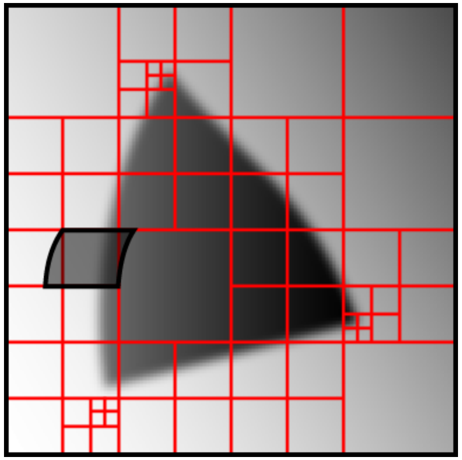

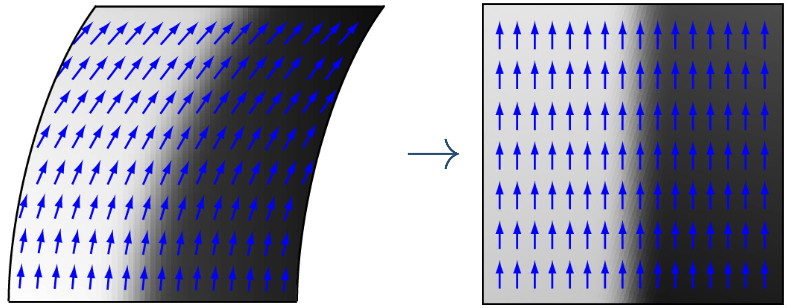

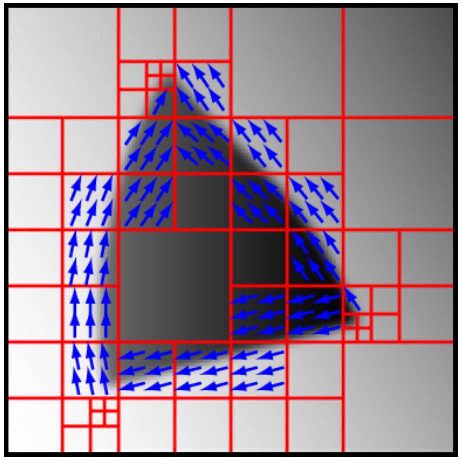

- Image $\mathbf{C}^\alpha - \mathbf{C}^\alpha$ simple piecewise.
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- Lagrangian approach: minimization of $\|f - f_M\|^2 + T^2 M \Longrightarrow$ thresholding in a fixed basis (easy) and best basis search (difficult).
- Hierarchical structure of the segmentation and additivity of the Lagrangian : Wickerhauser's best basis algorithm (CART).

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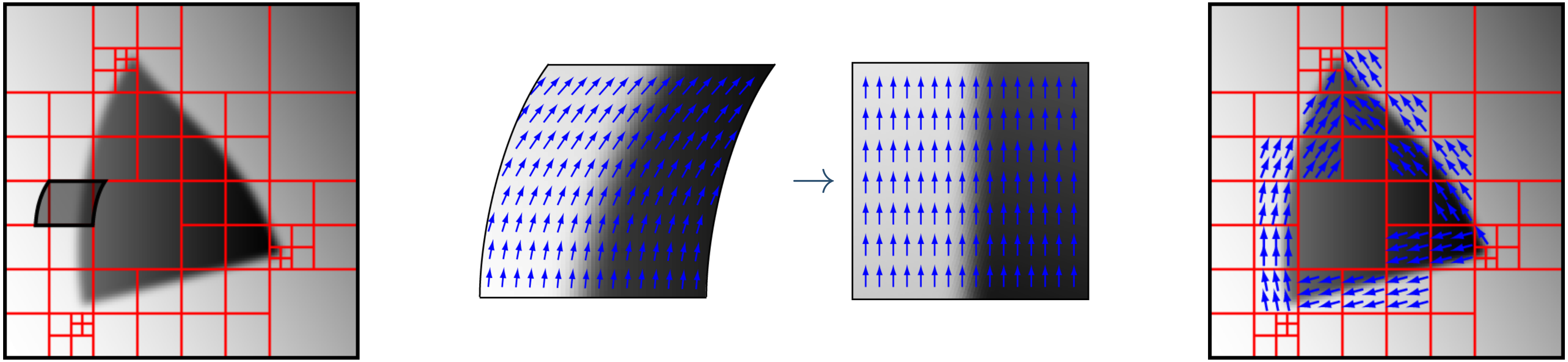

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- Hierarchical structure of the segmentation and additivity of the Lagrangian : Wickerhauser's best basis algorithm (CART).
- Exhaustive exploration of the geometries in each square.

# Bandlets and Multiscale Geometry

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- Bandlets: monoscale geometric representation.

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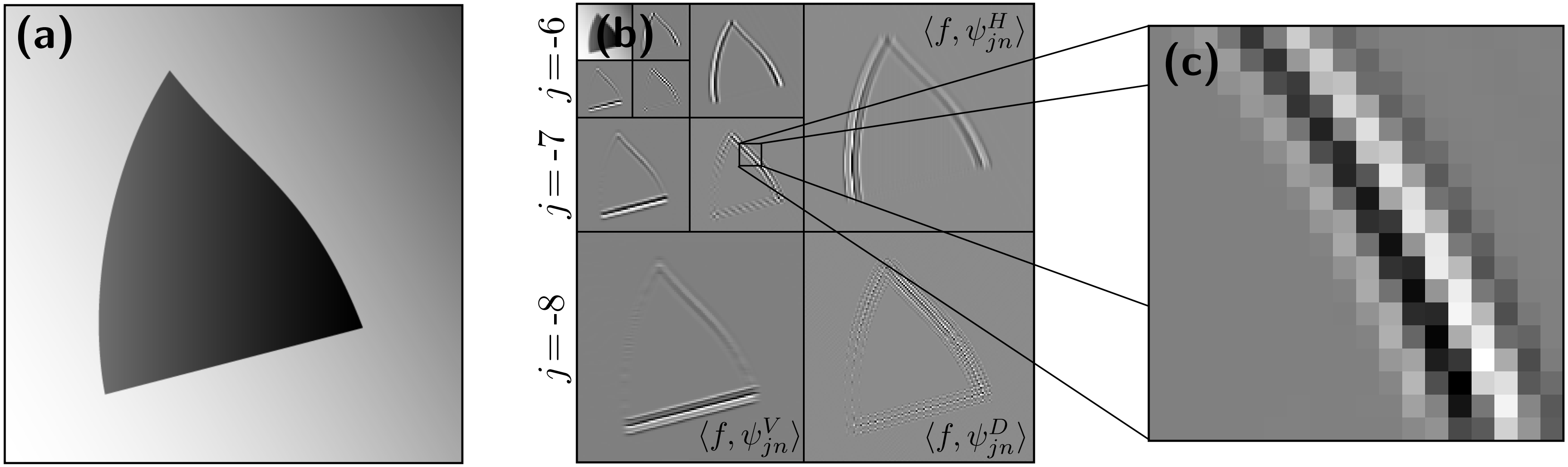


- Bandlets: monoscale geometric representation.
- Wavelets: good multiscale representation.

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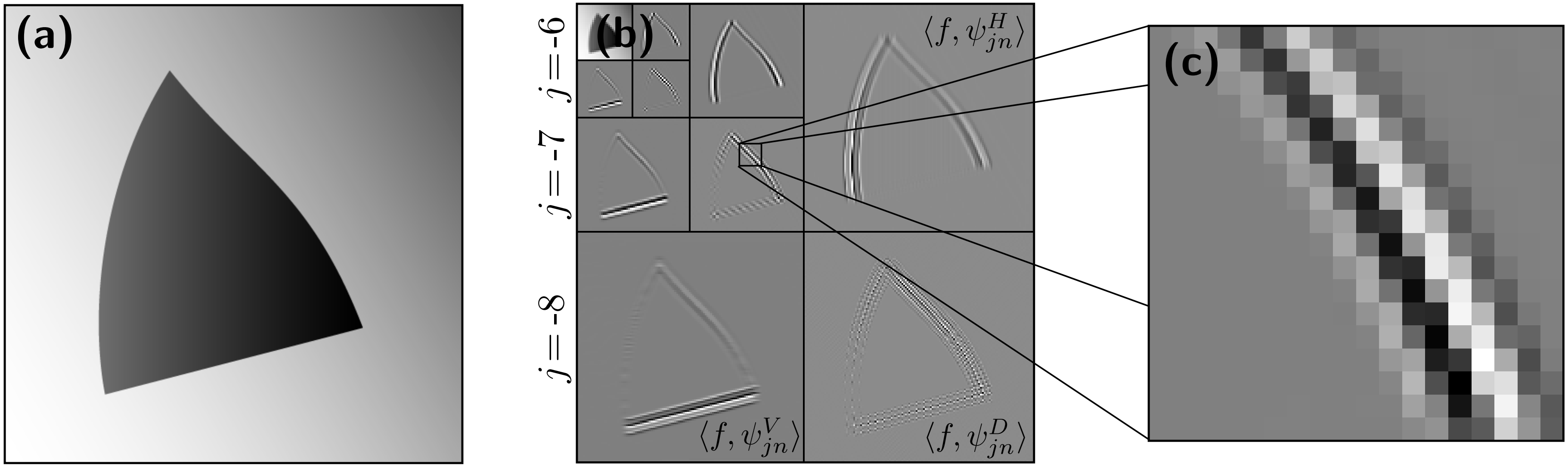


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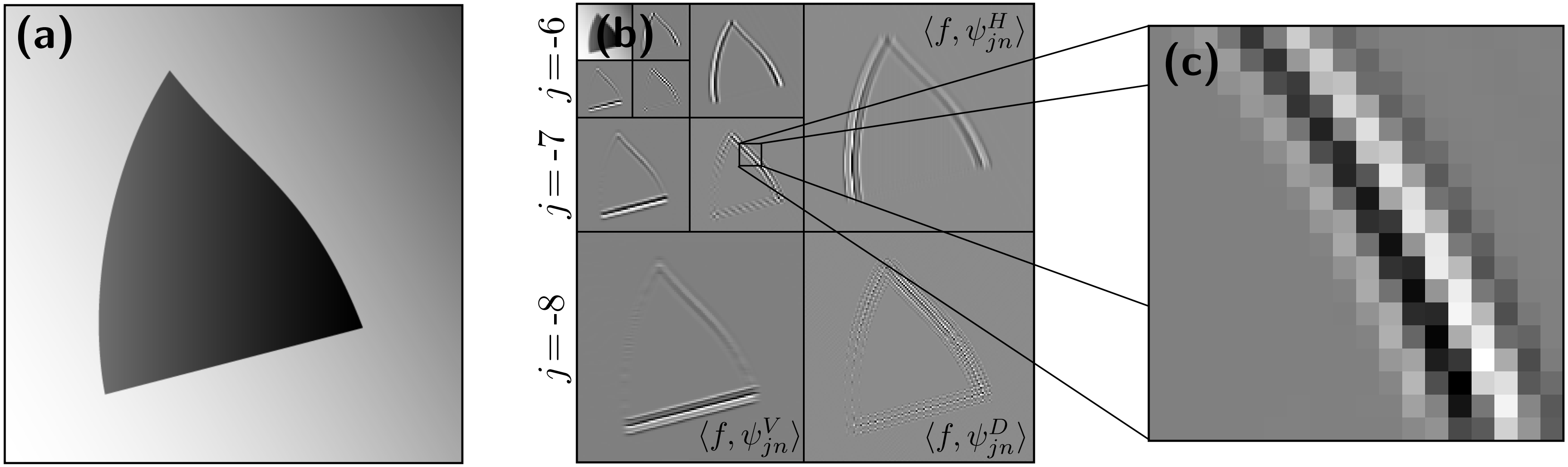


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- How to combine both?
- Bandlets on the wavelet coefficients!

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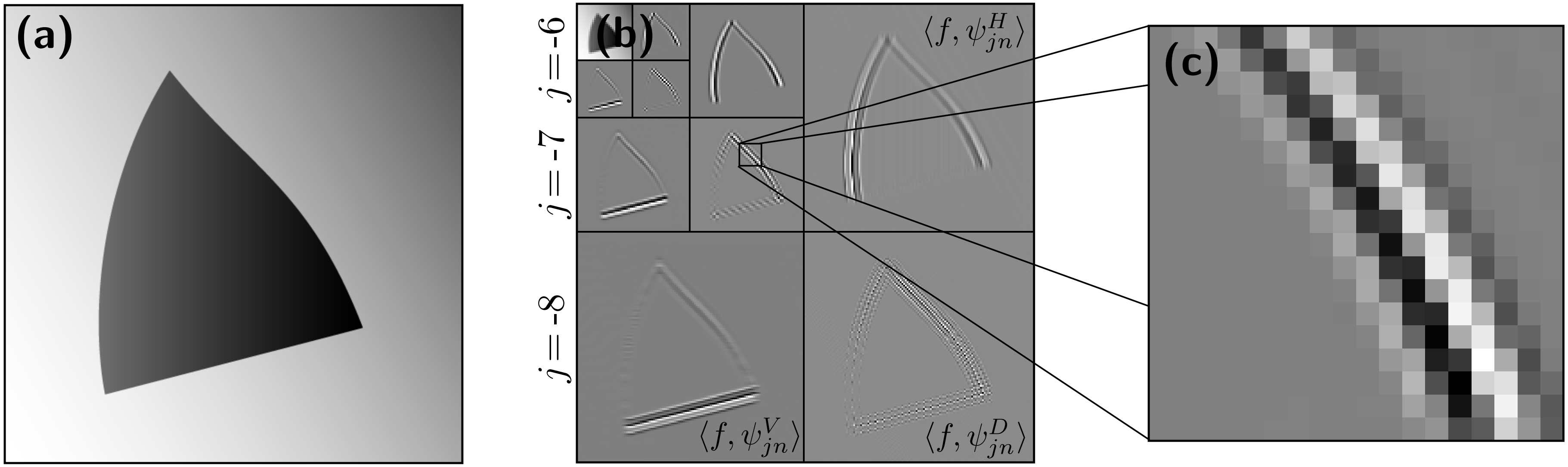


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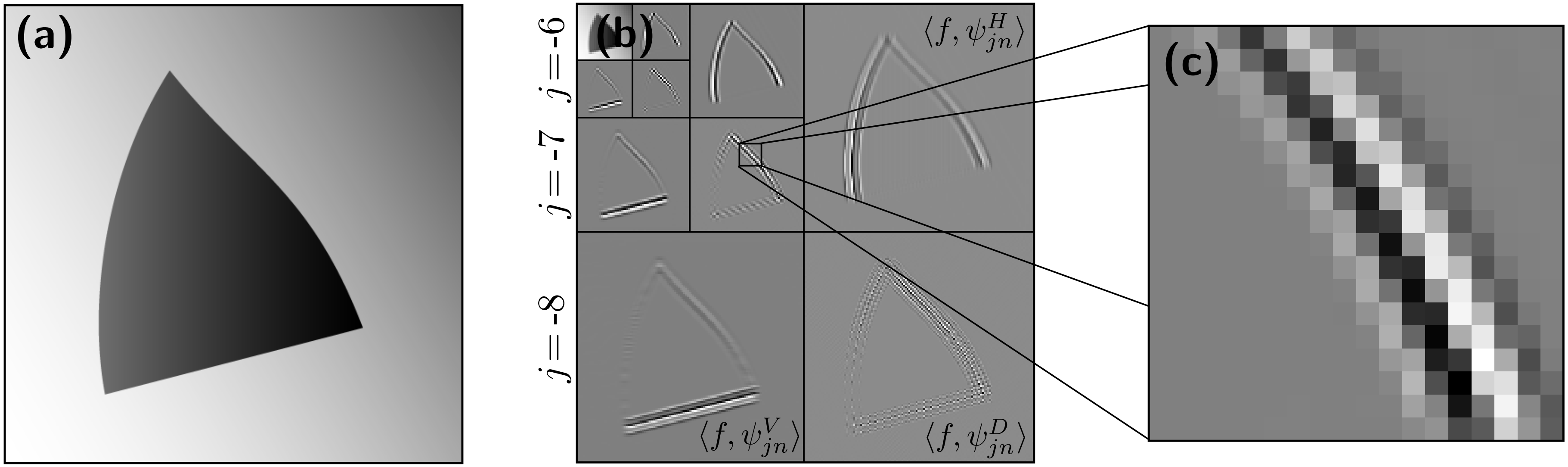


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- Similar ideas in JPEG2000, Edgeprint,...

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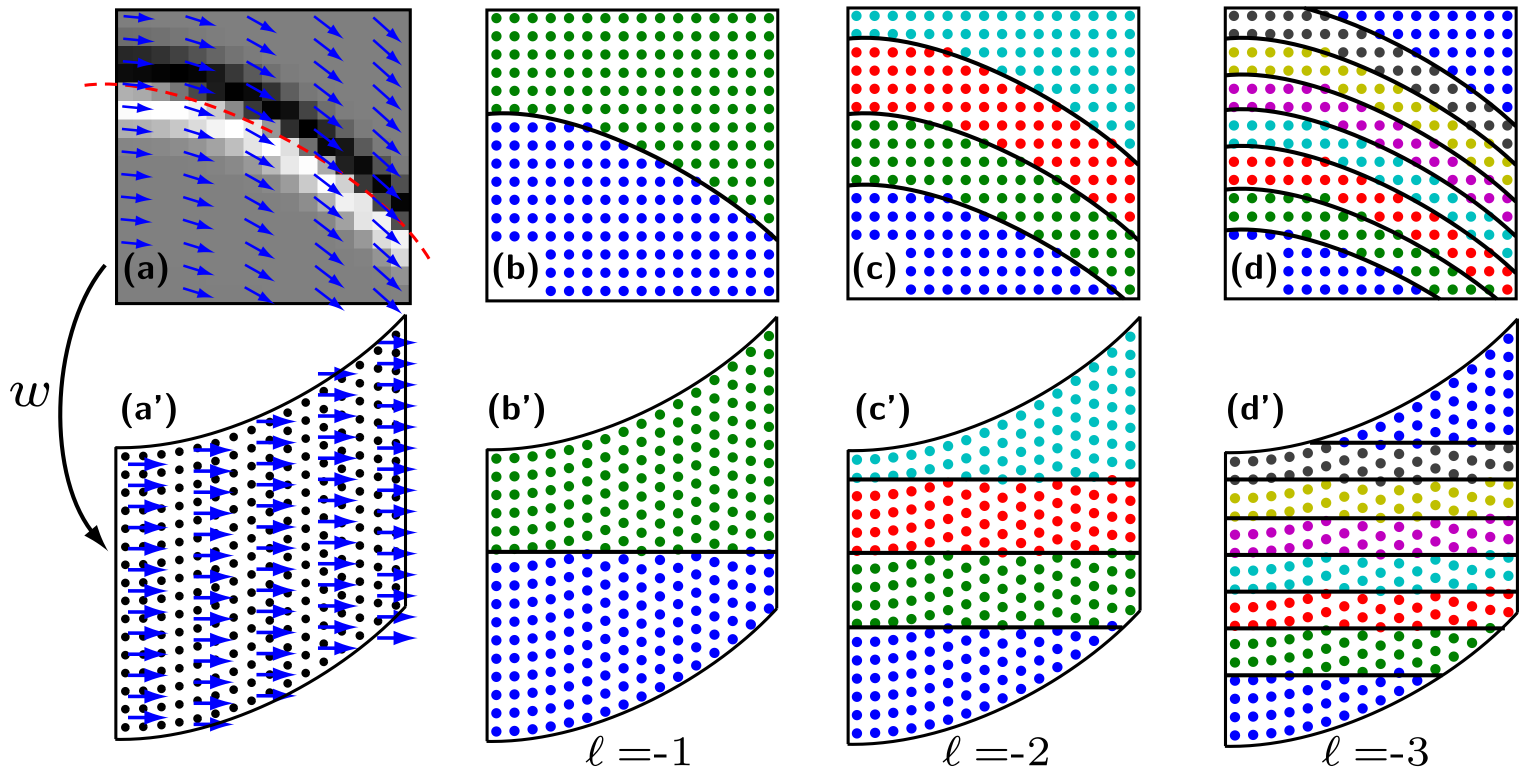


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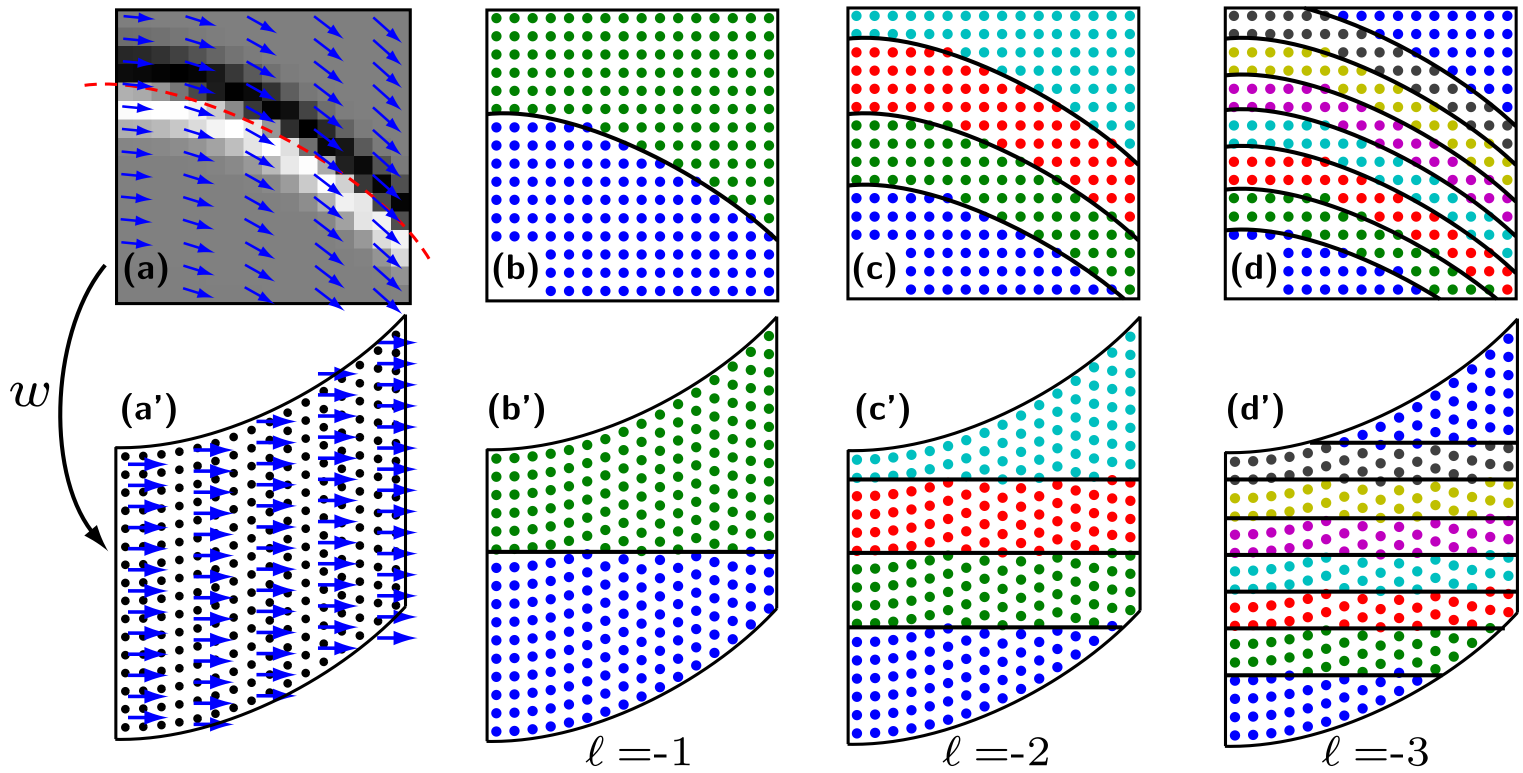


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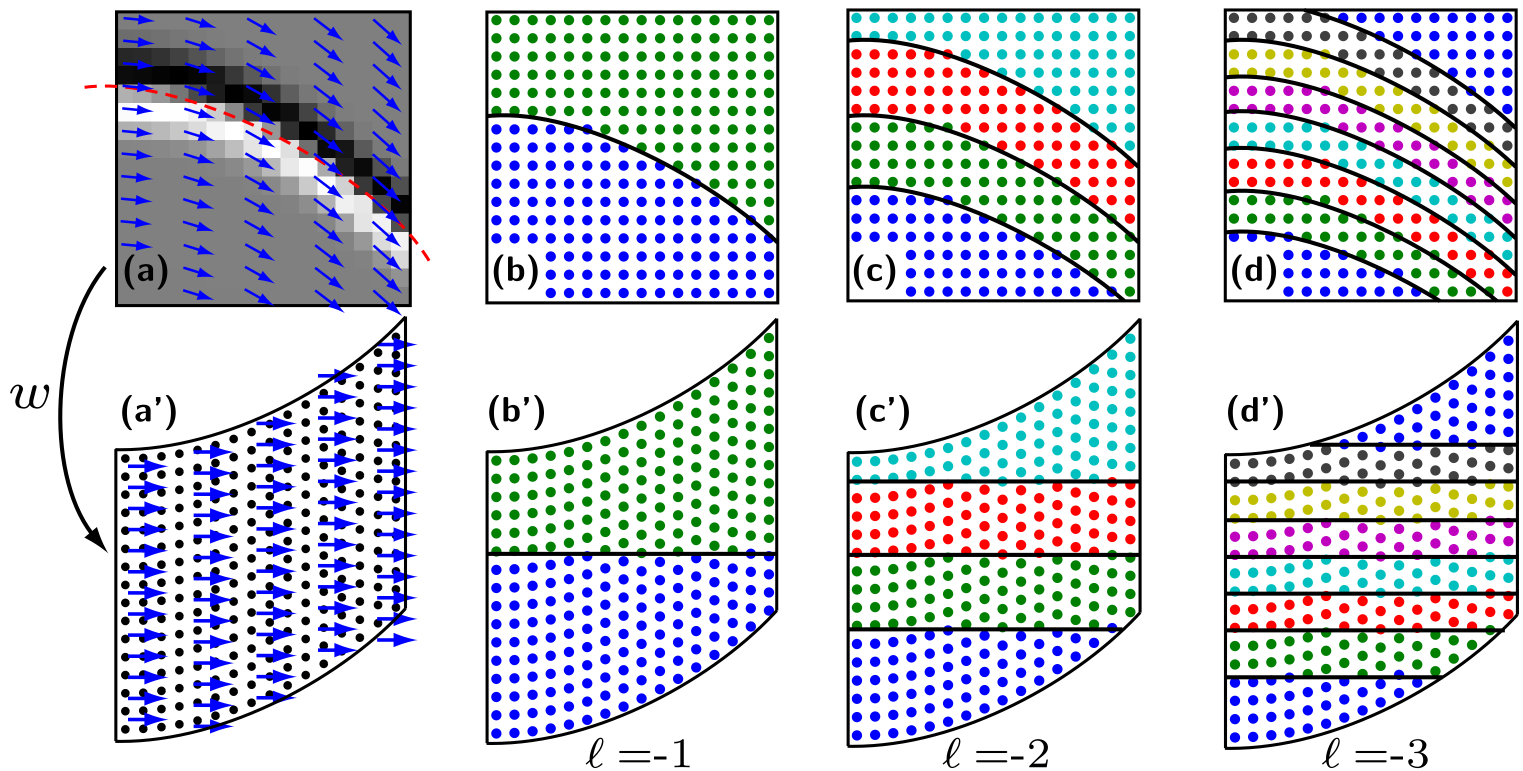


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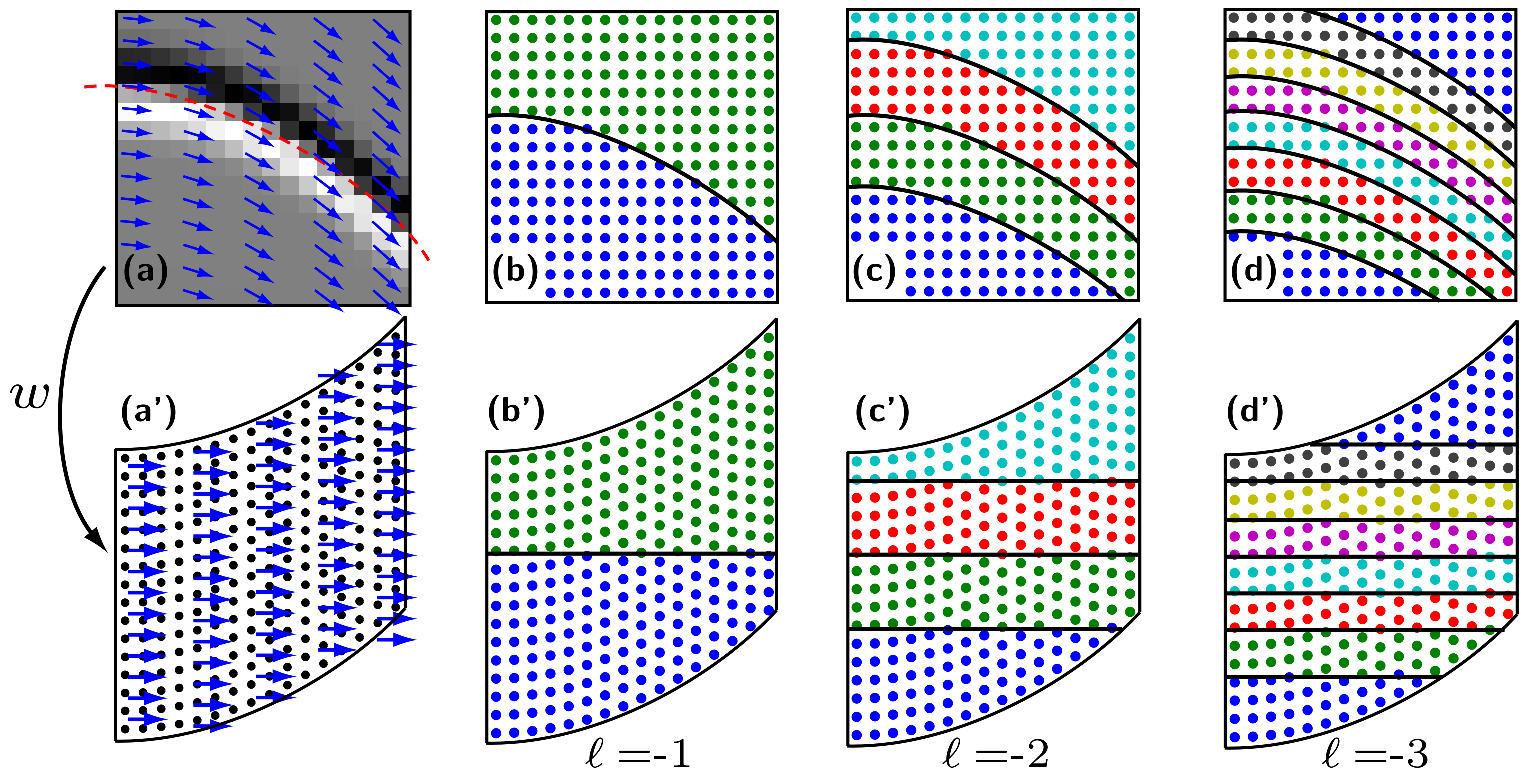


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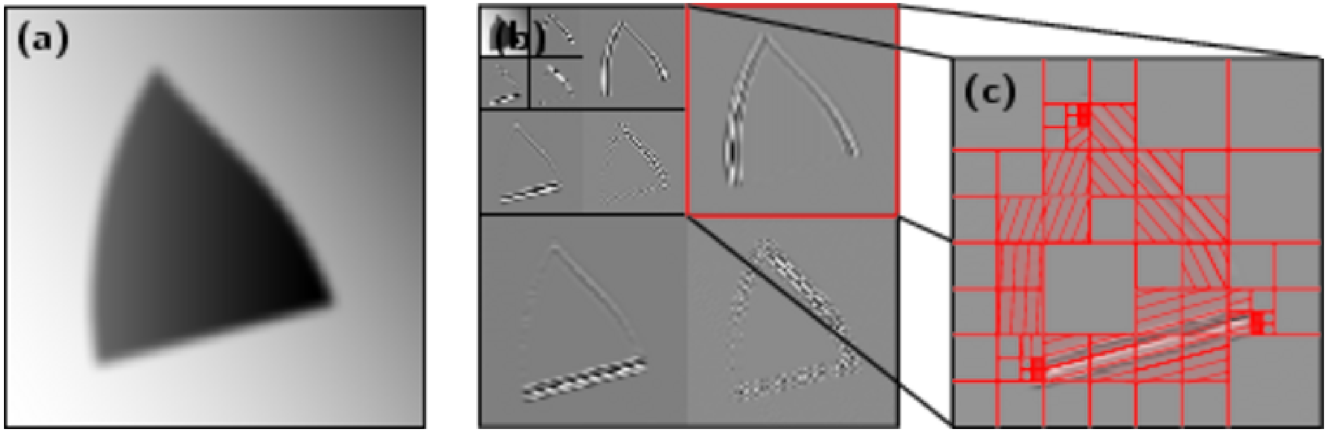


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  - dyadic segmentation of the subbands,
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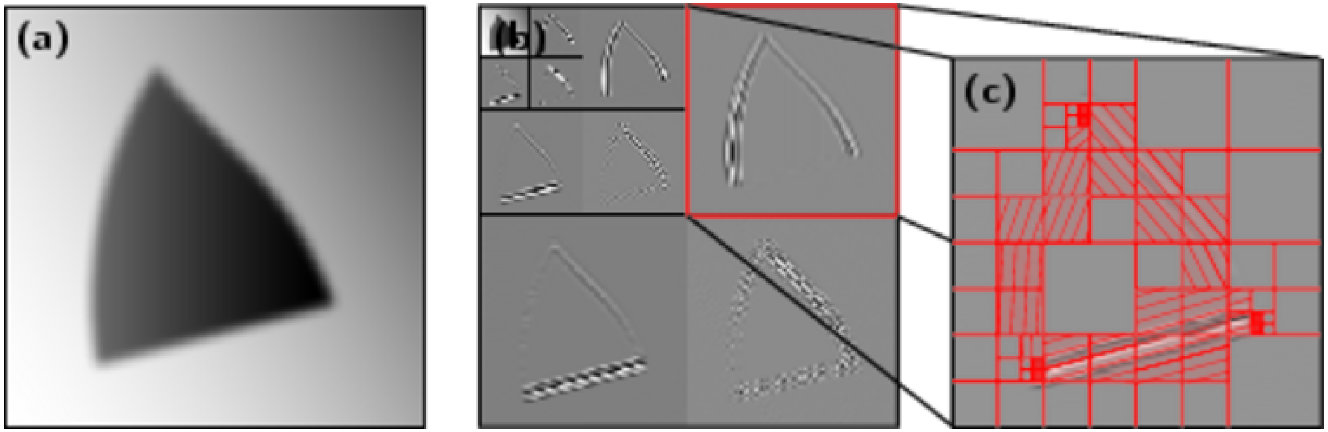


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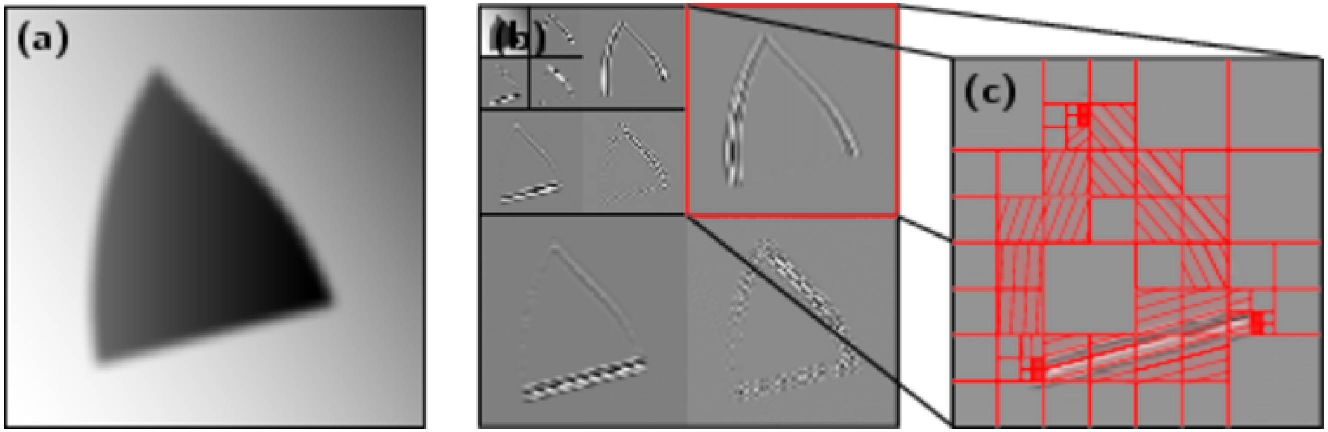


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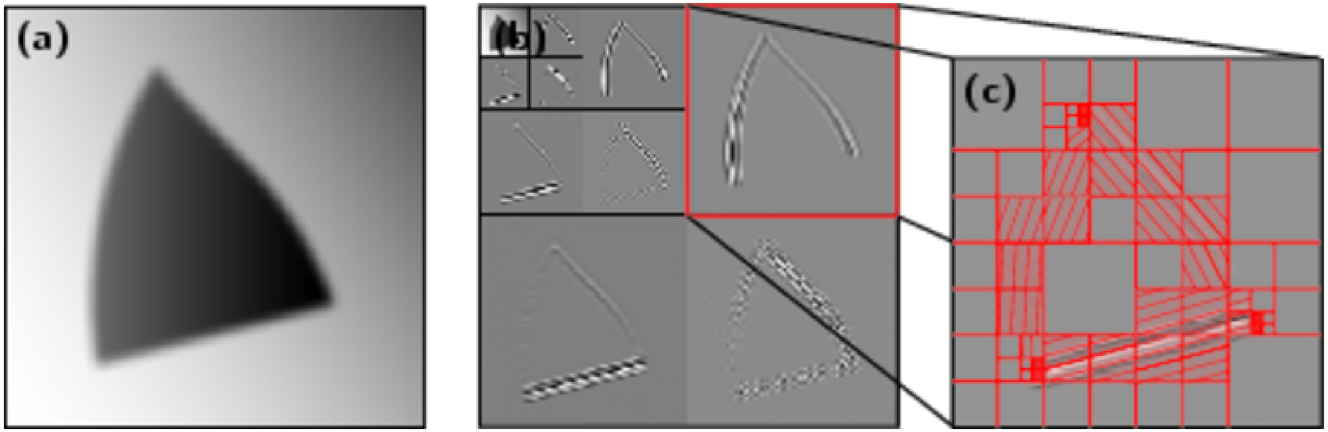


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- How to use these bases for estimation?

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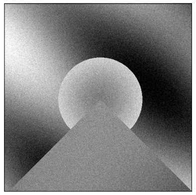

Estimation

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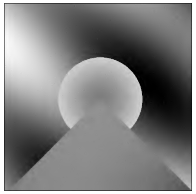

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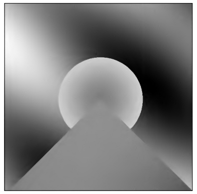

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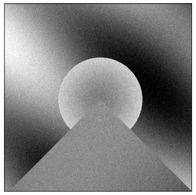

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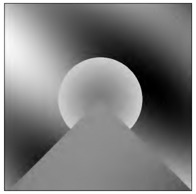

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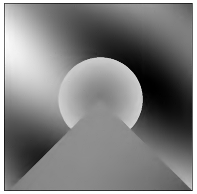

- Questions:
  - How is this working?
  - Why is it almost optimal for $\mathbf{C}^\alpha - \mathbf{C}^\alpha$ functions? (Minimax)
  - For which functions this method works efficiently? (Maxiset)

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- How to measure the performance of a given estimator? Minimax or Maxiset?

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- Key factor: Approximation properties of the basis!

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- Natural question: Is $\mathbf{C}^\alpha - \mathbf{C}^\alpha$ in $\mathcal{W}B^\alpha{}_{2/(2\alpha+1),2/(2\alpha+1)}$ ?

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- Big issue: how to threshold in the “best” bandlet basis without knowing $f$?

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- Suitable approach when more than one basis is considered.

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$$\Leftrightarrow \min_{m \in \mathcal{M}_\epsilon} \|f - P_m f\|^2 + T^2 \dim(m) \leq C \left(T^2\right)^{\frac{\beta}{\beta+1}}$$

$\Leftrightarrow \|f - f_M\|^2 \leq C M^{-\beta}$ plus linear approximation property

$$\Leftrightarrow f \in \mathcal{A}^{\beta}{}_{\kappa} \quad .$$

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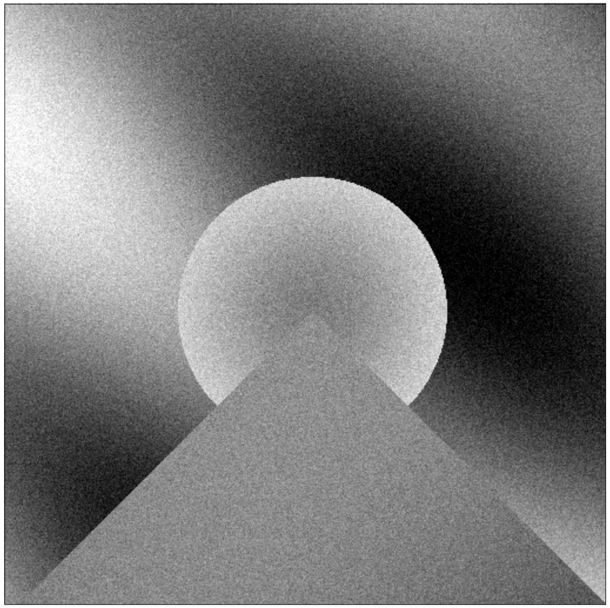
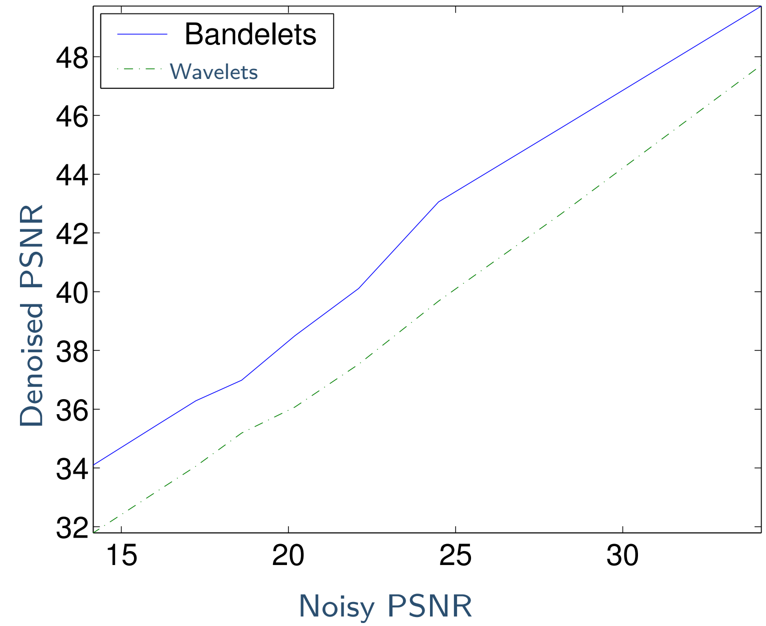
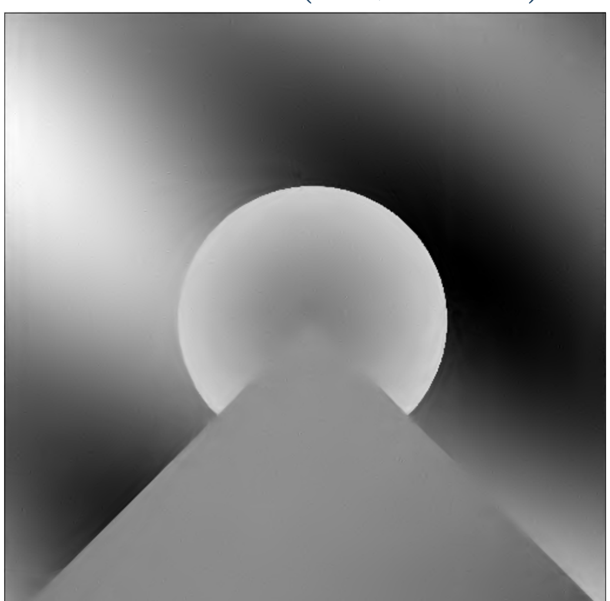
Noisy (20,19 dB)

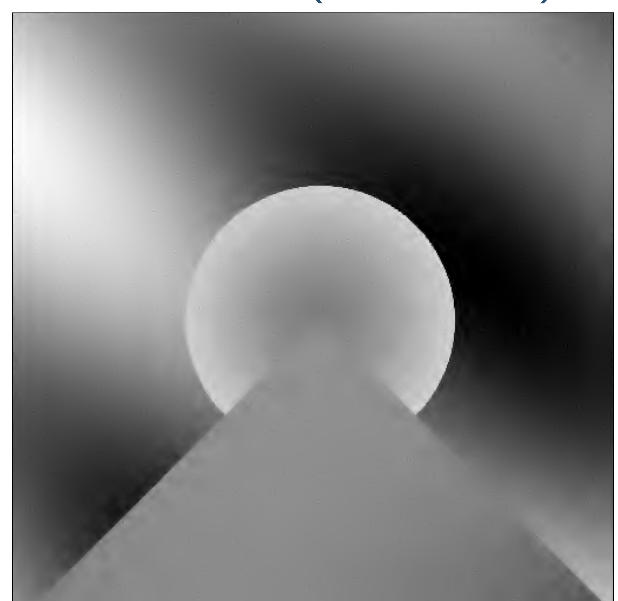





Bandlets (30,29 dB)



Wavelets (28,21 dB)



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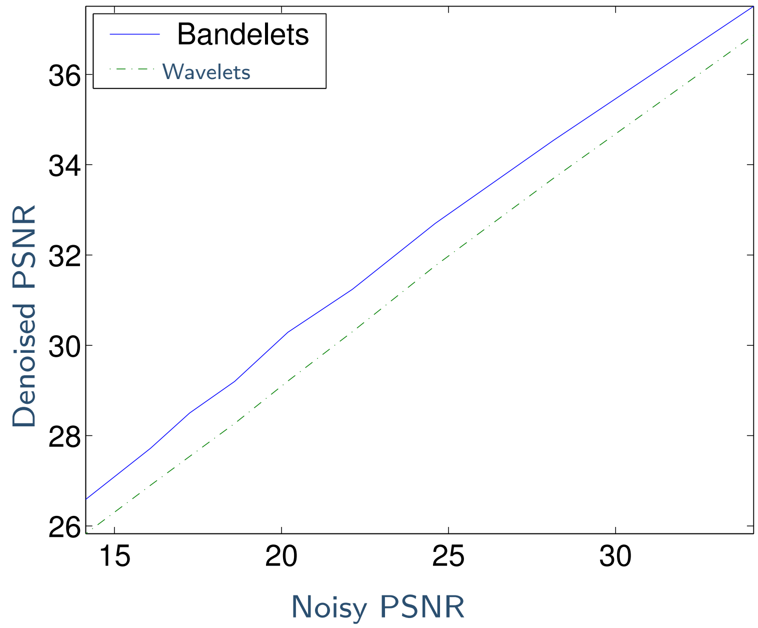


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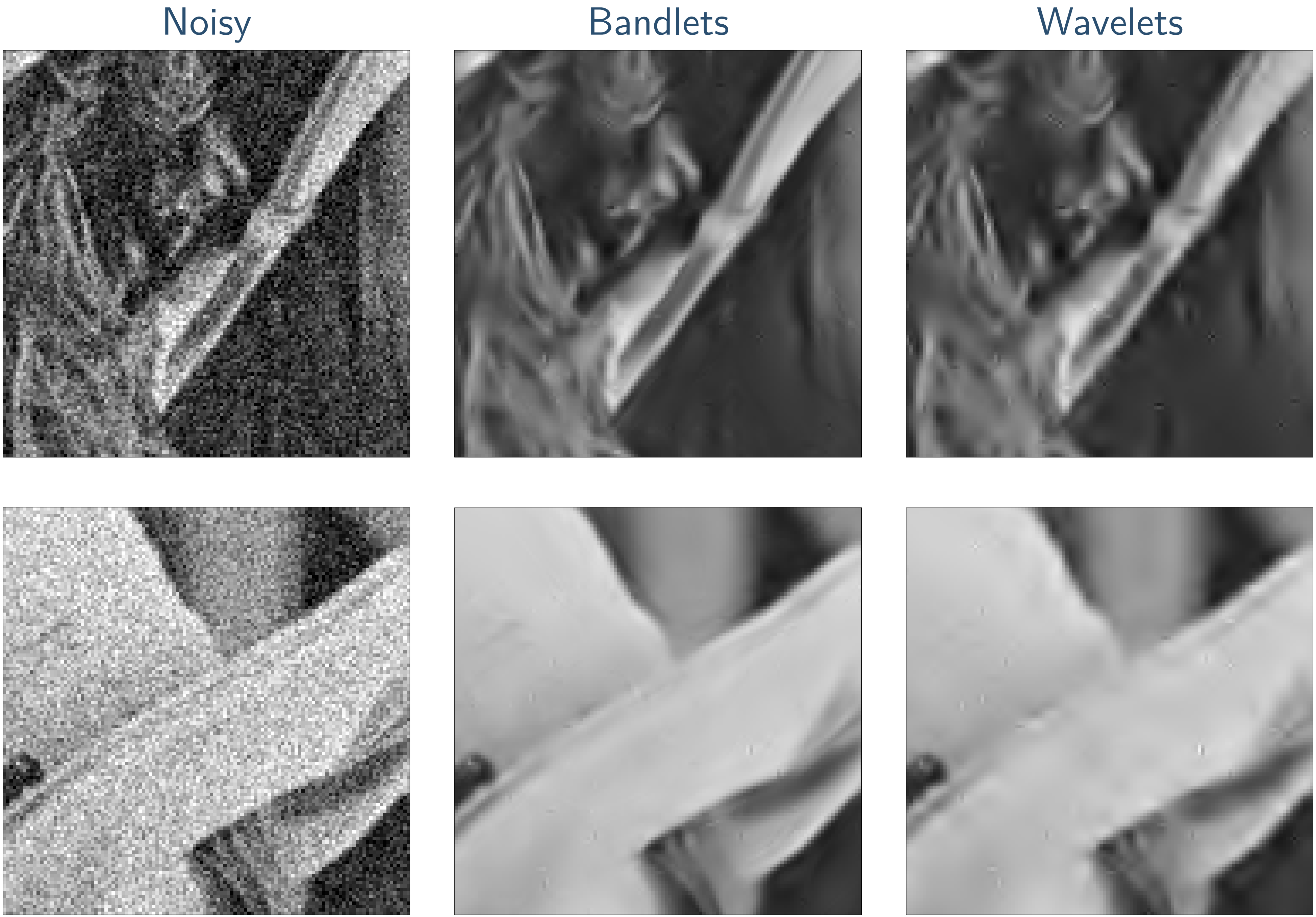
Noisy
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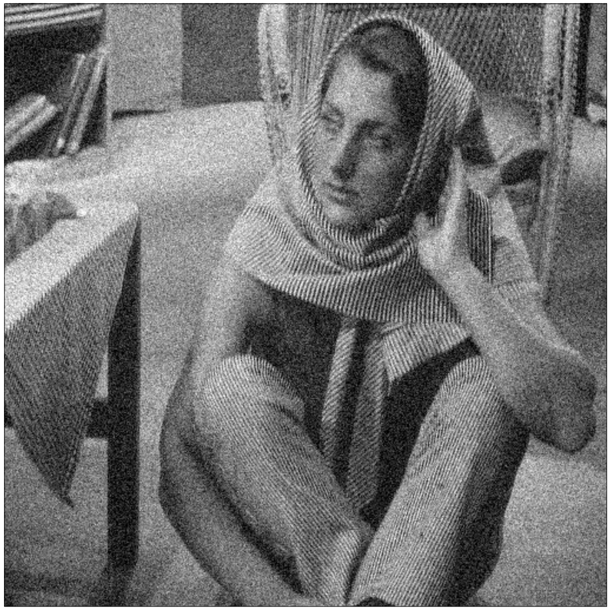
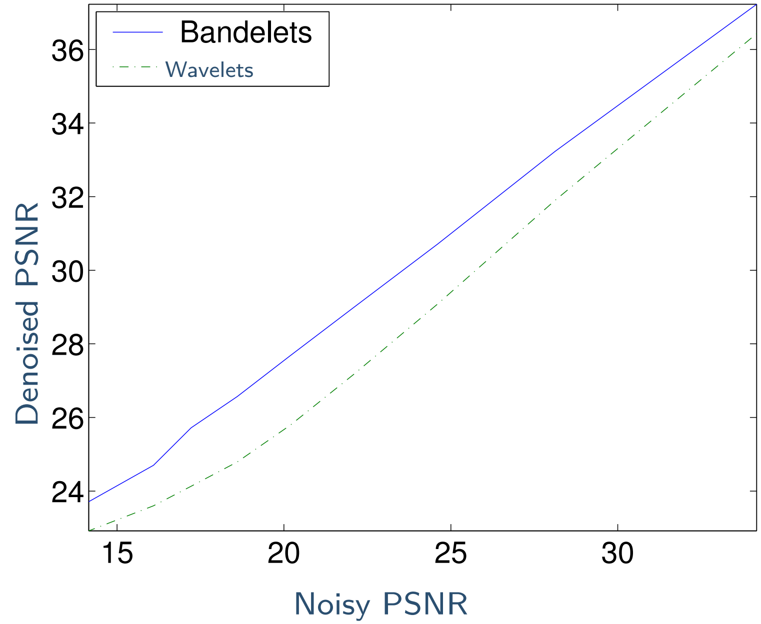
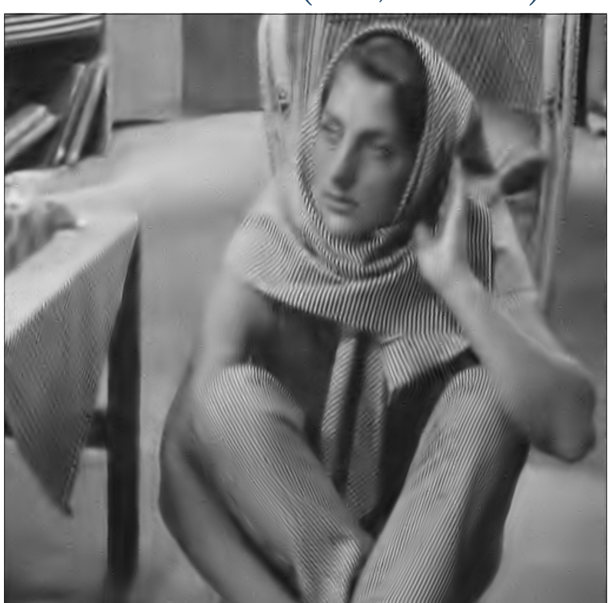
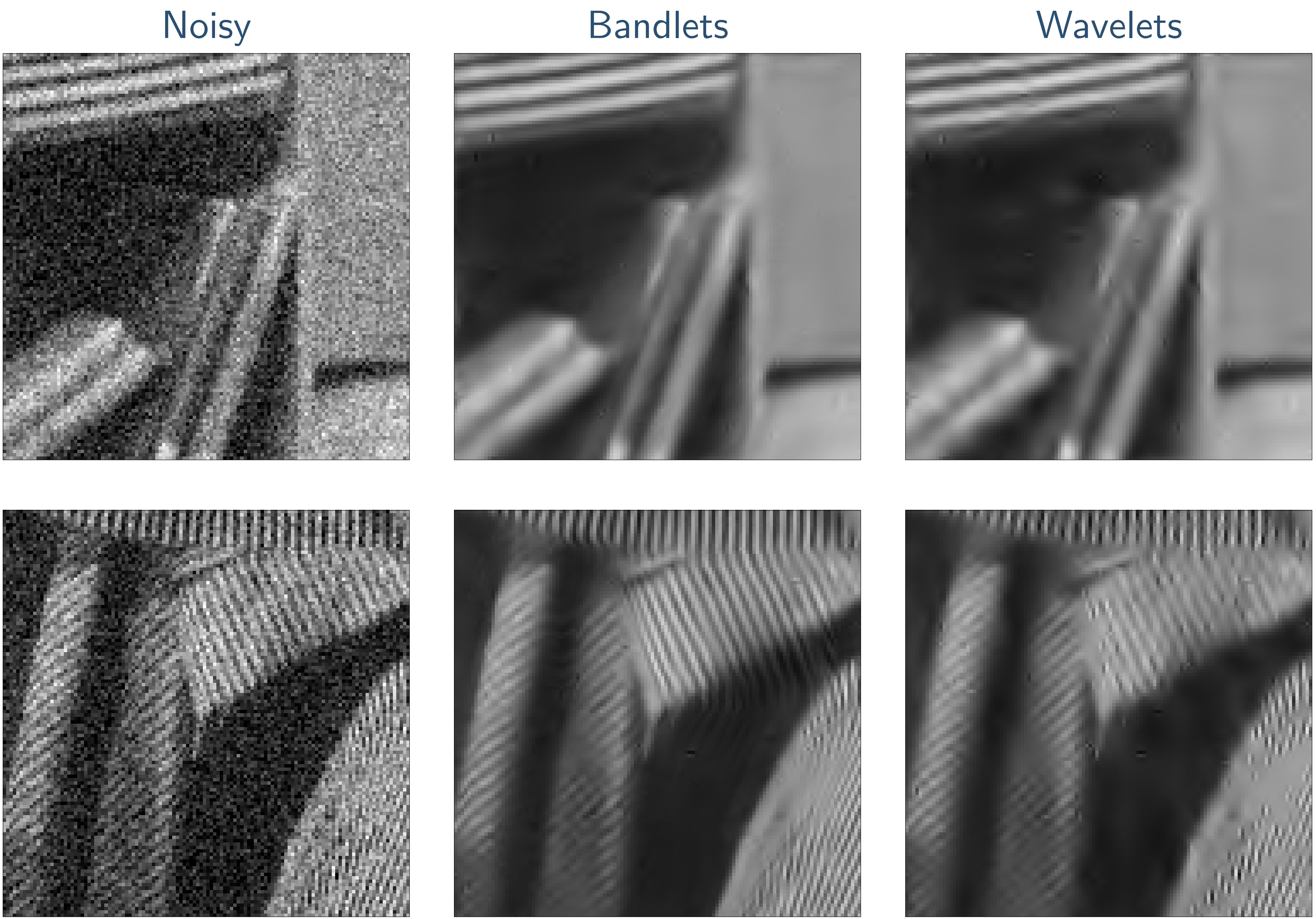





Bandlets (27,68 dB)



Wavelets (25,79 dB)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# Sketch of proof

- Model selection and Maxiset Theorems sketch of proof.
- $\epsilon^2 = \frac{1}{N}$
- Players:
  - Estimate: $F_S = \operatorname{argmin}_{m \in \mathcal{M}} \|Y - P_m Y\|^2 + \lambda \frac{\log N}{N} \dim(m)$
  - Model Selection Oracle:
    $f_O = \operatorname{argmin}_{m \in \mathcal{M}} \|f - P_m f\|^2 + \lambda \frac{\log N}{N} \dim(m)$
  - Maxiset Oracle: $f_{O'} = \operatorname{argmin}_{m \in \mathcal{M}} \|Y - P_m Y\|^2 + \lambda \frac{\log N}{4N} \dim(m)$
- Model selection:

$$E(\|f - F_S\|^2 + \lambda \frac{\log N}{N} \dim(m_F)) \leq C \left( \|f - f_O\|^2 + \lambda \frac{\log N}{N} \dim(m_O) \right) ??$$

- Maxiset:

$$\|f - f_{O'}\|^2 + \lambda \frac{\log N}{N} \dim(m_{O'}) \leq C E(\|f - F_S\|^2) \quad ??$$

- Model selection: general case but in probability.
- Maxiset: simple case (embedded models).

# Model Selection – 1

- Players:
  - Estimate: $F_S = \mathrm{argmin}_{m \in \mathcal{M}} \|Y - P_m Y\|^2 + \lambda \frac{\log N}{N} \dim(m)$
  - Model Selection Oracle:
    $f_O = \mathrm{argmin}_{m \in \mathcal{M}} \|f - P_m f\|^2 + \lambda \frac{\log N}{N} \dim(m)$
- Model selection: With large probability

$$\|f - F_S\|^2 + \lambda \frac{\log N}{N} \dim(m_F) \leq C \left( \|f - f_O\|^2 + \lambda \frac{\log N}{N} \dim(m_O) \right)$$

- Uniform noise control over all models $m$: Gaussian concentration inequality

$$P\left( \forall m, \quad \|P_m W\| \leq \sqrt{12 \log N \dim(m)} \right) \geq 1 - \frac{1}{N}$$

# Model Selection – 2

- By definition:
$$\|Y - F_S\|^2 + \lambda \frac{\log N}{N} \dim(m_F) \leq \|Y - f_O\|^2 + \lambda \frac{\log N}{N} \dim(m_O)$$
and using $\|Y-g\|^2 = \|Y-f\|^2 + 2\langle Y-f, f-g\rangle + \|f-g\|^2$
$$\|f - F_S\|^2 + \lambda \frac{\log N}{N} \dim(m_F) \leq \|f - f_O\|^2 + \lambda \frac{\log N}{N} \dim(m_O) + \frac{2}{\sqrt{N}}\langle W, f_O - F_S\rangle$$
- $|\langle W, f_O - F_S\rangle| \leq \|P_{m_O \cup m_F} W\| \, \|f_P - F_S\|$ .
- $\|f_O - F_S\| \leq \|f_O - f\| + \|f - F_S\| \leq 2(\|f - F_S\|^2 + \lambda \frac{\log N}{N} \dim(m_F))^{1/2}$ .
- Concentration inequality:
$$P\left(\forall m, \quad \|P_m W\| \leq \sqrt{12 \log N \dim(m)}\right) \geq 1 - \frac{1}{N}$$
- Thus with $P \geq 1 - \frac{1}{N}$,
$$\|P_{m_O \cup m_F} W\| \leq \sqrt{12 \log N (\dim m_O + \dim(m_F))}$$
$$|\langle W, f_O - F_S\rangle| \leq \sqrt{48/\lambda^2}(\|f - F_S\|^2 + \lambda \frac{\log N}{N} \dim(m_F)) \quad .$$

# Maxiset – 1

- Maxiset:

$$\|f - f_{O'}\|^2 + \lambda \frac{\log N}{N} \dim(m_{O'}) \leq CE(\|f - F_S\|^2)???$$

- No but

$$E(\|f - F_S\|^2) \leq C \left( \frac{\log N)}{N} \right)^{\frac{\beta}{\beta+1}}$$

$$\implies \min_{m \in \mathcal{M}_N} \|f - P_m f\|^2 + \log N \frac{\dim(m)}{N} \leq C \left( \frac{\log N}{N} \right)^{\frac{\beta}{\beta+1}}$$

$$\implies f \in \mathcal{A}^{\beta}$$

- Simple proof when the models are embedded ($m_1 \subset m_2$ or $m_1 \supset m_2$ for all $m_1, m_2$).
- General case much more complex...
- Thresholding is a simple extension of the embedded model case.

# Maxiset – 2

- We are going to prove that $\|f - F_S\|^2 \geq \|f - f_O\|^2$ !
- If $m_F \subset m_O$, $\|f - F_S\|^2 \geq \|f - P_{m_F} f\|^2 \geq \|f - f_O\|^2$.
- Otherwise $m_F \supset m_O$ and $\|f - F_S\|^2 = \|f - P_{m_F} f|^2 + \|P_{m_F} f - F_S\|^2 = \|f - P_{m_O} f|^2 + \|P_{m_O} W|^2 + \|P_{m_F \setminus m_O} W\|^2 - \|P_{m_F \setminus m_O} f\|^2$ .
- Recall that

$$\|Y - P_{m_F} Y\|^2 + \lambda \frac{\log N}{N} \dim(m_F) \leq \|Y - P_{m_O} Y\|^2 + \lambda \frac{\log N}{N} \dim(m_O)$$

$$\|f - P_{m_O} f\|^2 + \lambda \frac{\log N}{4N} \dim(m_O) \leq \|f - P_{m_F} f\|^2 + \lambda \frac{\log N}{4N} \dim(m_F)$$

- Thus

$$\|P_{m_F \setminus m_O} f\|^2 \leq \frac{1}{4} \|P_{m_F \setminus m_O} Y\|^2$$

$$\|P_{m_F \setminus m_O} f\|^2 \leq \frac{1}{2} \left( \|P_{m_F \setminus m_O} W\|^2 + \|P_{m_F \setminus m_O} f\|^2 \right)$$

which leads to

$$\|P_{m_F \setminus m_O} f\|^2 \leq \|P_{m_F \setminus m_O} W\|^2$$

# Maxiset – 3

- We prove that $\|f - f_O\|^2 \leq \|f - F_S\|^2$ and thus that

$$\|f - f_O\|^2 \leq E(\|f - F_S\|^2) \leq C \left( \frac{\lambda_N}{N} \right)^{\frac{\beta}{\beta+1}}$$

- We should now work on the dependency on $N$ to control the number of term and prove that $\|f - f_O\|^2 + \lambda \frac{\log N}{N} \dim(m_O) \leq C' \left( \frac{\lambda_N}{N} \right)^{\frac{\beta}{\beta+1}}$
- Using that $m_{O(N/2)}$ is not as efficient as $m_{O(N)}$ so that

$$\begin{aligned}\|f - f_{O(N)}\|^2 + \lambda \frac{\log N}{N} \dim(m_{O(N)}) &\leq \|f - f_{O(N/2)}\|^2 + \lambda \frac{\log N}{N} \dim(m_{O(N/2)}) \\ &\leq \frac{1}{2} \|f - f_{O(N/2)}\|^2 \\ &\quad + \frac{1}{2} \left( \|f - f_{O(N/2)}\|^2 + \lambda \frac{\log N}{N/2} \dim(m_{O(N/2)}) \right)\end{aligned}$$

- Slightly more tricky because of the $\log N$...
- Conclusion obtained by recursion.